

## Lecture 14 (March 21, 2016)

### Boundedness and Ultimate boundedness

Show boundedness of solutions even when no eq. pt. at origin.

Motivated Example.  $\dot{x} = -x + 8 \sin t$ ,  $x(t_0) = a > 8 > 0$

has no eq. pt. Solution is

$$x(t) = e^{-(t-t_0)} a + 8 \int_{t_0}^t e^{-(t-\tau)} \sin \tau d\tau$$

$$\rightarrow \|x(t)\| \leq e^{-(t-t_0)} a + 8 (1 - e^{-(t-t_0)})$$

and so  $\|x(t)\| \leq a$ ,  $\forall t \geq t_0$ . (ind. of  $t_0$ )

$\nwarrow$  too conservative

Further, let  $b$  be s.t.  $8 < b < a$ . Then,

$$\|x(t)\| \leq b, \quad \forall t \geq t_0 + \log \frac{a-8}{b-8} \quad e^{-(t-t_0)} \leq e^{-\ln \frac{a-8}{b-8}} = \frac{b-8}{a-8}$$

$b$  is ultimate bound.

Lyapunov analysis

$$\Rightarrow \|x(t)\| \leq \frac{b-8}{a-8} (a-8) + 8 = b$$

Let  $V = \frac{x^2}{2}$ . Then  $\dot{V} = -x^2 + x 8 \sin t \leq -x^2 + 8 |x|$

- $\dot{V}$  is not negative definite near the origin. However, negative definite outside set  $\{x : |x| \leq 8\}$ .
- Let  $c > 8^2/2$ . Solutions starting in set  $\{V(x) \leq c\}$  remains there for all future time, since  $\dot{V} < 0$  on  $V=c$ .
- So solutions are uniformly bounded.
- Further, if we pick  $\varepsilon$  s.t.  $8^2/2 < \varepsilon < c$ , then  $\dot{V} < 0$  in set  $\{\varepsilon < V < c\}$ , so solutions go to  $\{V < \varepsilon\}$  and stay there.
- Solution is uniformly ultimately bounded with ultimate bound

$$V = \frac{x^2}{2} \leq \varepsilon \Rightarrow |x| \leq \sqrt{2\varepsilon} \quad \text{Diagram: } -c \leftarrow \varepsilon \leftarrow -\frac{8^2}{2} \leftarrow \frac{8^2}{2} \leftarrow \varepsilon \leftarrow c$$

Def Solutions of  $\dot{x} = f(t, x)$  are

1) uniformly bounded: if  $\exists$  a positive constant  $c$ , independent of  $t_0$ , and for every  $a \in (0, c)$ ,  $\exists \beta = \beta(a) > 0$ , independent of  $t_0$ , s.t.

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \quad \forall t \geq t_0. \quad (*)$$

2) globally uniformly bounded: if  $(*)$  holds for arbitrary large  $a$ .

3) uniformly ultimately bounded with ultimate bound "b": if  $\exists b > 0$ ,  $c > 0$ , independent of  $t_0 \geq 0$ , & for each  $a \in (0, c)$ ,  $\exists T = T(a, b) \geq 0$ , independent of  $t_0 \geq 0$ , s.t.

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b \quad \forall t \geq t_0 + T \quad (**)$$

4) globally uniformly ultimately bounded with ultimate bound "b": if  $(**)$  holds for arbitrary large  $a$ .

Theorem 4.18. Let  $D \subseteq \mathbb{R}^n$  contains  $x=0$  &  $V: [0, \infty) \times D \rightarrow \mathbb{R}$  be c' st.

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \quad (1)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \quad \forall \|x\| \geq M > 0 \quad (2)$$

$\forall t \geq 0$ , and  $\forall x \in D$  where  $\alpha_1, \alpha_2 \in K$  and  $W_3$  is cont. & pos. def.

Take  $r > 0$  s.t.  $B_r \subset D$  and suppose that  $M < \alpha_2^{-1}(\alpha_1(r))$ .

Then,  $\exists \beta \in K$  and for any initial state  $x(t_0)$  satisfying

$\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$ ,  $\exists T \geq 0$  (dep on  $x(t_0)$  &  $M$ ) s.t. solution satisfies

$$+\begin{cases} \|X(t)\| \leq \beta(\|X(t_0)\|, t-t_0) & \forall t_0 \leq t \leq t_0 + T \\ \|X(t)\| \leq \underbrace{\alpha_1^{-1}(\alpha_2(M))}_{\in K} & \forall t \geq t_0 + T \end{cases} \quad (\text{u.b})$$

(as  $M \rightarrow 0$ , bound  $\rightarrow 0$ )

If  $D = \mathbb{R}^n$  and  $\alpha_1 \in K_\infty$ , then  $(+)$  holds for any  $x(t_0)$  and no restriction on  $M$ .

Main application studying stability of Perturbed systems.

Example (Duffing's equation)  $m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A \cos wt$

$$\begin{aligned} X_1 &= y & \dot{X}_1 &= X_2 \\ X_2 &= \dot{y} & \dot{X}_2 &= -(1+X_1^2)X_1 - X_2 + M \cos wt \\ & & \dot{X}_2 &= -\frac{c}{m}X_2 - \frac{k}{m}X_1 - \frac{ka^2}{m}X_1^3 + \frac{A}{m} \cos wt \\ & & c = m = k, a = 1, \frac{A}{m} = M \end{aligned}$$

$M=0 \Rightarrow (0,0)$  : eq. pt & g.a.s with Lyapunov function:

$$V(X) = X^T P X + \frac{1}{2} X_1^4 \quad (\text{see Example 4.6})$$

$$P = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

when  $M>0$ , use Thm 4.18 with  $V(X)$  as above.

$V(X)$  : positive definite & radially unbounded

$\Rightarrow$  By Lemma 4.3.  $\exists \alpha_1, \alpha_2 \in \mathbb{K}_{\infty}$  that satisfies (1) of Thm 4.18.

$$\dot{V} = -X_1^2 - X_1^4 - X_2^2 + (X_1 + 2X_2)M \cos wt \leq -\|X\|_2^2 - X_1^4 + M\sqrt{5}\|X\|_2$$

$$(X_1 + 2X_2) = (1 \ 2) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \leq \|(1 \ 2)\|_2 \|X\|_2 = \sqrt{5}\|X\|_2$$

$$\xrightarrow{\text{split } -\|X\|_2^2} \dot{V} \leq -(1-\theta)\|X\|_2^2 - X_1^4 - \theta\|X\|_2^2 + M\sqrt{5}\|X\|_2 \quad 0 < \theta < 1$$

$$\text{For } \forall \|X\|_2 \geq \underbrace{\frac{M\sqrt{5}}{\theta}}_{\mu}, \quad \dot{V} \leq \underbrace{-(1-\theta)\|X\|_2^2 - X_1^4}_{-W_3(X)}$$

Hence,  $\dot{V}$  satisfies (2) of Thm 4.18.

$\Rightarrow$  The solutions are g.u.u.b

Next step: calculate the ultimate bound

$$\underbrace{\lambda_{\min}(P)\|X\|_2^2}_{\alpha_1(\|X\|)} \leq X^T P X \leq V(X) \leq X^T P X + \frac{1}{2}\|X\|_2^4 \leq \underbrace{\lambda_{\max}(P)\|X\|_2^2 + \frac{1}{2}\|X\|_2^4}_{\alpha_2(\|X\|)}$$

$$\text{Thus the ultimate bound is } b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\alpha_2(\mu)}{\lambda_{\min}}} = \sqrt{\frac{\lambda_{\max} M^2 + \mu^4}{\lambda_{\min}}}$$

## Input to state stability (ISS)

### Motivation.

Similar idea to ultimate boundedness except think of perturbation as input. I.e., consider  $\dot{x} = f(t, x, u)$  with bounded input  $u(t)$ . Suppose unforced system  $\dot{x} = f(t, x, 0)$  has g.u.a.s eq. pt at origin. What can be said of behavior of system in presence of bounded input?

**Example.** Suppose we have a Lyapunov function  $V$  for unforced system and we look at  $\dot{V}$  for forced system. Can we show that  $\dot{V}$  is negative outside a ball of radius  $\mu$ ? This would then allow us to use the Lyapunov theory for boundedness & ultimate boundedness.

### Goal of ISS.

- 1) develop a toolkit of concepts for studying systems via decompositions.
- 2) quantification of system response to external signals.
- 3) unification of state-space and I/O stability theories not discussed yet

For linear, time-invariant systems,  $X=0$  is g.u.a.s iff  $A$  is Hurwitz.

Also,  $A$  Hurwitz gives automatically all reasonable notions of I/O stability.  
bounded input  $\rightarrow$  bounded output

inputs converging to zero  $\rightarrow$  outputs converging to zero etc

g.a.s of  $X=0$  for nonlinear systems does not give this kind of results.

**Example.**  $\dot{x} = -x + (1+x^2)u$ . When  $u=0$ ,  $X=0$  is g.a.s.

But solutions diverge for some inputs that converge to zero:

Let  $u(t) = (2t+2)^{-\frac{1}{2}}$  and  $X(0) = \sqrt{2}$ .

Then  $X(t) = (2t+2)^{\frac{1}{2}}$  which diverges.



**Def.** Consider the system  $\dot{x} = f(t, x, u)$ , where  $u$  is bounded & piecewise continuous and  $f$  is piecewise continuous in  $t$  & locally Lipschitz in  $x+u$ .  $\dot{x} = f(t, x, u)$  is ISS if  $\exists \beta \in KL$  and  $\gamma \in K$  st. for any  $x(t_0)$  and any bounded input  $u(t)$ ,  $x(t)$  exists  $\forall t \geq t_0$  and satisfies

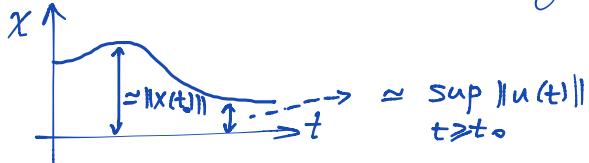
$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \right)$$

(i) bounded input  $\Rightarrow$  bounded state

(ii)  $x(t)$  is ultimately bounded by  $\gamma \left( \sup_{t \geq t_0} \|u(t)\| \right)$

(iii)  $u(t) \rightarrow 0$  as  $t \rightarrow \infty \Rightarrow x(t) \rightarrow 0$  as  $t \rightarrow \infty$   
ISS  $\Rightarrow x=0$  (when  $u=0$ ) is g.u.o.s.

ISS combines overshoot and asymptotic behavior:



In case of linear system get  $\|x(t)\| \leq \beta(t) \|x(0)\| + \gamma \sup_{t \geq 0} \|u(t)\|$   
where  $\beta(t) = \|e^{At}\| \rightarrow 0$  and

$$\gamma = \|\beta\| \int_0^\infty \|e^{sA}\| ds < \infty$$

**Theorem 4.19.** Let  $V: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $c^1$  function st.

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \quad (1)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W_3(x) \quad \forall \|x\| \geq \rho(\|x\|) > 0 \quad (2)$$

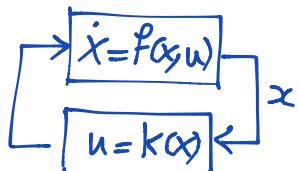
$\forall (t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$ ,  $\alpha_1, \alpha_2 \in K_\infty$ ,  $\rho \in K$ ,  $W_3$ : cont & pos. def

Then, the system is ISS with  $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$ .

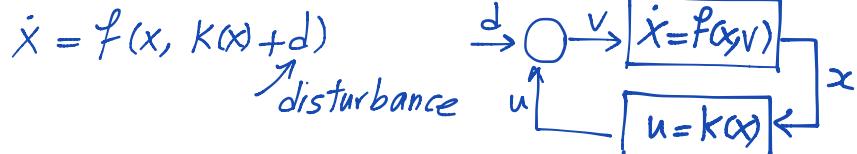
**Prob.** (uses global version of Thm 4.18) Note  $x(t)$  only depends on  $u(\tau)$  for  $t_0 \leq \tau \leq t$ .

Notion of ISS arose originally to formulate & answer the following question:

suppose  $K$  chosen so that for  $\dot{x} = f(x, K(x))$ ,  $x=0$  is g.a.s.  
what is the effect of input disturbances (activator dist./noise,  
modeling uncertainty)?



Study new system



Disturbances can destabilize system:

$$\text{Example. } \dot{x} = f(x, u) = x + (x^2 + 1)u$$

Let  $\tilde{u}$  be feedback linearization defined by  $u = \frac{\tilde{u}}{1+x^2}$ .

Then,  $\dot{x} = x + \tilde{u}$ . choose  $\tilde{u} = -2x \Rightarrow \dot{x} = -x$

i.e.  $u = K(x) = \frac{-2x}{1+x^2} \Rightarrow \dot{x} = f(x, K(x)) = -x$  which is g.a.s.

Now suppose there is disturbance  $d$ :  $\dot{x} = -x + (x^2 + 1)d$

As shown before, this system has solutions that diverge to  $\infty$  even for inputs  $d$  that converge to zero! Moreover, the constant input  $d=1$  implies solutions explodes in finite time.

Thus,  $K(x) = \frac{-2x}{1+x^2}$  is not a good feedback law.

Suppose we choose instead  $\tilde{K}(x) = \frac{-2x}{1+x^2} - x$ . Then,  $\dot{x} = f(x, \tilde{K}(x) + d)$

becomes  $\dot{x} = -2x - x^3 + (1+x^2)d$ .  $x=0$  is g.a.s and the system is ISS.

(For large  $x$ ,  $-x^3$  dominates  $(1+x^2)$  for all bounded  $d(\cdot)$  and this prevents the state from getting very large.)

One of the main features of LSS is that it behaves well under composition.

A cascade of LSS is LSS.  $\rightarrow \boxed{x} \rightarrow \boxed{z}$

consider  $\dot{x}_1 = f_1(t, x_1, x_2)$  (1) }  
 $\dot{x}_2 = f_2(t, x_2)$  (2) } (Σ)  $f_2 \xrightarrow{x_1} \boxed{z}$

Suppose  $x_1=0$  is g.u.a.s for  $\dot{x}_1 = f_1(t, x_1, 0)$  } what about  
and  $x_2=0$  is g.u.a.s for  $\dot{x}_2 = f_2(t, x_2)$  } (0,0) for cascade system

Theorem 4.7. If (1) with  $X_2$  as input is LSS and  $x_2=0$  is  
g.u.a.s. Then  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  is g.u.a.s for (2).